

RESIDUAL GENERIC ERGODICITY OF PERIODIC GROUP EXTENSIONS OVER TRANSLATION SURFACES

DAVID RALSTON AND SERGE TROUBETZKOY

ABSTRACT. Continuing the work in [13], we show that within each stratum of translation surfaces, there is a residual set of surfaces for which the geodesic flow in almost every direction is ergodic for almost-every periodic group extension produced using a technique referred to as *cuts*.

1. INTRODUCTION

A translation surface is a real 2-dimensional manifold with conical singularities equipped with an atlas for which transition functions are translations. The simplest example of a compact translation surface is the 2 dimensional torus. It is a classical result that the straight-line flow on the torus satisfies a dichotomy: for rational directions it is completely periodic, while for irrational directions it is uniquely ergodic. There has been considerable work in extending and generalizing this result to other translation surfaces; here we list the major achievements. Veech has shown that this dichotomy hold for any compact translation surface whose group of affine diffeomorphism is a lattice in $SL(2, \mathbb{R})$ [17]. Such surfaces are now known as *Veech surfaces*, this class includes all regular polygons with an even number of sides, with parallel sides identified, as well as compact square-tiled translation surfaces and other examples. Kerckhoff, Masur, and Smillie have shown that on any compact translation surface, the geodesic flow in almost every direction is uniquely ergodic [9]. Avila and Forni have shown that for almost every compact translation surface of genus at least 2, for almost every direction the geodesic flow is in fact weak-mixing [3].

On the other hand, much less is known for noncompact translation surfaces. For aperiodic infinite translation surfaces, recurrence is known for generic translation surfaces in certain classes [15, 16]. A bit more is known for periodic infinite translation surfaces. Hubert, Hooper, and Weiss constructed an example of an infinite translation surface for which the geodesic flow is ergodic in almost every direction [7]. One of the most famous infinite translation surfaces arises from the classical Ehrenfest wind-tree model, introduced by Paul and Tatiana Ehrenfest in 1912 [5]. Hubert, Lelièvre, and Troubetzkoy, and Avila and Hubert proved the recurrence of the geodesic flow for periodic wind-tree model [8, 2] and the diffusion rate was calculated by Delecroix, Hubert, and Lelièvre to be $2/3$ [4]. Fraćzek and Ulcigrai have shown that the geodesic flow for wind-tree model is not ergodic for almost all directions [6]. In [13] we showed that for square tiled translation surfaces which have a single cylinder direction, almost every infinite group extensions of geodesic flows produced by cuts is ergodic in almost every direction.

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In this article we extend this result to the following: there is a residual set of surfaces for which the geodesic flow in almost every direction is ergodic for generic periodic group extensions produced by cuts. The techniques used are the same ones as in [13]: essential values and approximation by purely periodic directions.

2. BACKGROUND AND DETAILS OF THE CONSTRUCTION

A translation surface is a Riemannian surface M , a finite or countable set of points $\Sigma := \{x_1, x_2, \dots\} \subset M$ (the singularity set) and an open cover of $M \setminus \Sigma$ by sets $\{U_\alpha\}$, together with charts $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^2$ such that for all α, β with $U_\alpha \cap U_\beta \neq \emptyset$ the map $\phi_\alpha \circ \phi_\beta^{-1}$ is a translation. The Euclidean metric and Lebesgue measure on \mathbb{R}^2 may be pulled back to M via these charts except at singular points $x_i \in \Sigma$ with cone angle $2\pi(d_i + 1)$ for $d_i \in \mathbb{Z}^+$; this set Σ is finite if M is compact. We denote by μ the pull-back of Lebesgue measure to M (if $\mu(M) < \infty$, we normalize to that $\mu(M) = 1$). The collection of all compact translation surfaces with n singularities of cone angles $2\pi(d_i + 1)$ for $i = 1, 2, \dots, n$ (where the d_i are monotone decreasing by convention) is referred to as a *stratum*, and denoted by $\mathcal{H}(d_1, \dots, d_n)$. Any compact translation surface M may be viewed (in a non unique way) as a polygon (not necessarily convex) in \mathbb{R}^2 , where pairs of parallel congruent sides are identified (see e.g. [19, §12]). The precise parameterization of strata that we will use is that of the space of *zippered rectangles*, which we will define later.

When M is a k -fold cover of the torus ramified over a point, then M is called *square-tiled*. On each flat surface M we also choose a marked point p in a specified chart U_α , and we require only $p \notin \Sigma$.

On M , with the flat metric lifted from the charts, geodesics that do not go through singularities project to straight lines in the plane in a fixed direction, and such geodesics are either periodic or simple (do not intersect themselves). If the geodesic in direction θ starting at $x \in M$ does not intersect with Σ then we define the geodesic flow $\varphi_t^\theta(x)$ to the point obtained after moving in the direction θ for a time t starting at x . If θ is fixed, we may simply write $\varphi_t(x)$. If the geodesic hits a singularity, then the geodesic flow $\varphi_t(x)$ is defined up to this time. The geodesic flow preserves the surface area measure μ . For a survey on translation surfaces, see [19, 20].

We may lift generic points and geodesics from the plane to M ; more precisely consider the marked point $p \in U_\alpha$ and $p' := \phi_\alpha(p) \in \mathbb{R}^2$. Let $x \in \mathbb{R}^2$ be arbitrary, and consider the line segment connecting p' to x . Let ℓ denote the length of this segment, note that part of this segment is in $\phi_\alpha(U_\alpha)$. Lift this part to U_α and, if possible (i.e. if the procedure does not run into a singularity), continue this line in M until it is of length ℓ , in which case the line segment from p' to x on \mathbb{R}^2 *lifts* to a unique geodesic on M . This lift exists for almost all $x \in \mathbb{R}^2$.

Definition 1. A *collection of cuts* γ is a finite collection of pairs of parallel congruent line segments in the plane. Specifically, let $N \in \mathbb{Z}^+$, and for each $i = 1, 2, \dots, N$, let

$$(\mathbf{x}_i^+, \mathbf{x}_i^-) := ((x_i^+, y_i^+), (x_i^-, y_i^-)) \in \mathbb{R}^2 \times \mathbb{R}^2$$

be a pair of points in the plane, $\zeta_i \in \mathbb{S}^1$, and $\ell_i \in \mathbb{R}^+$. Denote by γ_i^+ and γ_i^- the pair of geodesics beginning at the two points \mathbf{x}_i^\pm , in direction ζ_i and of length ℓ_i . All possible finite collections of cuts are parameterized by the σ -finite measure

space

$$(1) \quad \Gamma := \bigcup_{N=1}^{\infty} (\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}^+)^N,$$

equipped with Lebesgue measure. We will generally not distinguish between the points \mathbf{x}_i^\pm in the plane and their lifts to a surface M , and similarly for the segments γ_i^\pm , allowing us to more easily speak of placing the same collection of cuts on two different surfaces.

Given a compact translation surface M , we will form *skew products* over M in the following manner: let G be a locally compact Abelian group with translation-invariant metric $\|\cdot\|$, whose identity element we denote e_G , and let γ be a collection of N cuts. Lift each segment in γ to M ; if some γ_i^\pm encounters a singularity on M then the lift will not be unique, but there will be only finitely many continuations, and for the time being we may simply make a choice of continuation. Let $\{g_1, g_2, \dots, g_N\}$ be a finite set in G . Define $f : M \rightarrow G$ to take value g_i on the lift of each γ_i^+ , value $-g_i$ on the lift of γ_i^- , and value e_G elsewhere. Wherever two (or more) lifts of γ_i cross, define the function at these points to be the sum of the function on those segments. Given a point $x \in M$ and a direction θ which is not equal to any ζ_i , denote

$$(2) \quad S_t(x) := S_t(x, M, \theta) := \sum_{\substack{0 \leq s < t \\ f(\varphi_s^\theta(x)) \neq e_G}} f(\varphi_s^\theta(x)),$$

and with M and θ fixed,

$$(3) \quad S_t^{-1}(B_\epsilon(g)) := \{x \in M : \|S_t(x) - g\| < \epsilon\}.$$

Then the transformation $\tilde{\varphi}_t$, from $\tilde{M} = M \times G$ to itself, is given by

$$\tilde{\varphi}_t(x, g) = (\varphi_t(x), g + S_t(x)),$$

and it is this continuous time flow which we study; $\tilde{\varphi}_t$ preserves the measure $\tilde{\mu} = \mu \times \nu$, where ν is the Haar measure on G . See Figure 1 to see how one single cut γ_i^+ might be appear on a surface with a *single-cylinder direction*, i.e. a direction θ so that the flow on M in this direction is completely periodic, with all points having the same period h , such that there is an open, connected set which is invariant under the flow in this direction, contains no singularities, and is of full measure. The surface M may thus be represented as a strip with the elements of Σ restricted to the sides parallel to the flow. Geodesics which connect two singularities are called *saddle connections*.

In order for the flow on \tilde{M} to be ergodic in direction θ , it is necessary that the projection of the cuts to a segment in a direction transverse to θ has an average value of e_G (for example, projected to the base of a single-cylinder). Note that by assigning the values $\pm g_i$ on parallel segments of equal length, we have ensured that this condition is always met.

Definition 2 (Generically ergodic extensions). Suppose that M is a compact translation surface so that for any locally compact Abelian group G , for almost every collection of cuts and almost every $\theta \in \mathbb{S}^1$, the flow $\tilde{\varphi}_t^\theta$ on \tilde{M} is ergodic for any choice of values $\{g_1, g_2, \dots, g_N\}$ on the cuts which generate a dense subgroup of G . Then M will be said to *produce generically ergodic extensions via cuts*. Let \mathcal{E} be

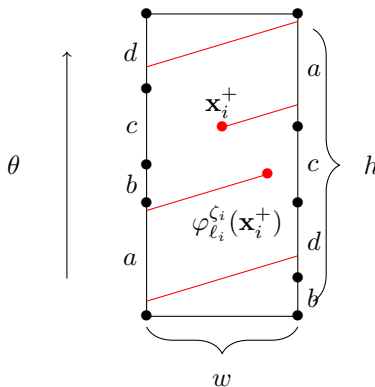


FIGURE 1. The single-cylinder representation for a surface M with completely periodic flow in direction θ with period h , with the flow represented as vertical. The length $w = \mu(M)/h$ is called the width of the cylinder. Parallel saddle connections of equal lengths are identified by the letters a through d , and the top and bottom are identified. A sample cut on the surface is the lift of γ_i^+ , beginning at \mathbf{x}_i^+ (the lift of (x_i^+, y_i^+) to M), in direction ζ_i and of length ℓ_i . A corresponding cut, parallel and of equal length, is the lift of γ_i^- .

the set of translation surfaces M which produce generically ergodic extensions via cuts.

One central result of [13] is that if M is a square-tiled surface with a single-cylinder direction, then $M \in \mathcal{E}$. It is of interest to know how typical this property is for translation surfaces in general. In this article we are not concerned with answering this question regarding measure-theoretic generic M , but rather we answer the question of *topological* genericity. In a Baire space Y , a set $A \subseteq Y$ is called G_δ if it may be written as the countable intersection of open, dense sets, and A is called *residual* if it contains a G_δ set.

We will work in the space of *zippered rectangles*; this space was introduced by Veech [18], and we follow the presentation of Zorich [20, §5.7]. Fix integers $d_i \geq 1$, and the corresponding stratum $\mathcal{H}(d_1, \dots, d_n)$. Surfaces in this stratum have n singular points, their genus g satisfies $\sum_{i=1}^n d_i = 2g - 2$. Fix a translation surface M in this stratum and a direction θ without saddle connections, this direction will be called *vertical*, the perpendicular direction will be called *horizontal*. For the rest of this article the term *translation surface* will correspond to the pair (M, θ) . Suppose that X is a horizontal section. The first return map to X is always an interval exchange transformation. We suppose furthermore that X satisfies Convention 2 of [20]: the first return of the vertical flow to X has the minimal possible number

$$N := \sum (d_j + 1) + 1 = 2g + n - 1$$

of subintervals under exchange. Furthermore we will always place the left endpoint of X at a singularity, the convention leaves a discrete choice for the position of the right endpoint. The first return map decomposes M into rectangles: these rectangles together with their identifications are referred to as *zippered rectangles*.

See Figure 2 for a convex polygon with vertical flow in which we induce the first-return flow to a horizontal segment, and Figure 3 for the rearrangement of the figure as a collection of rectangles.

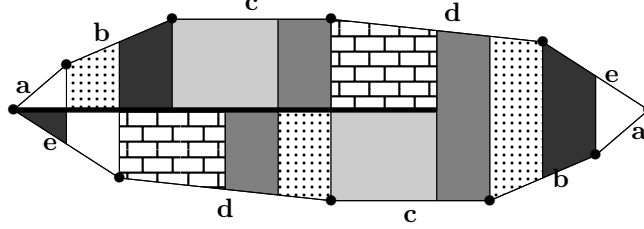


FIGURE 2. A polygon (e.g. a decagon) with pairs of parallel sides defines a translation structure (side identifications are marked). By defining a segment transverse to the vertical flow, we define a first-return map to this segment.

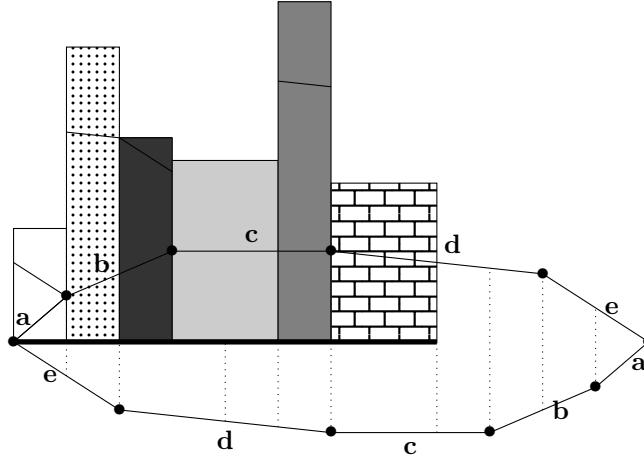


FIGURE 3. The first-return flow to the segment in Figure 2 may be rearranged as a collection of rectangles rather than an arbitrary polygon, where the tops of rectangles are identified to the bottom of the segment, inducing an *interval exchange transformation*.

The coordinates of zippered rectangle start with the usual coordinates of interval exchange: the widths $\lambda := (\lambda_1, \dots, \lambda_N)$ of the rectangles (intervals), and the permutation $\pi \in S_N$ describing the order of the exchanged intervals. Next we need the heights of $h := (h_1, \dots, h_N)$ the rectangles, and $a := (a_1, \dots, a_N)$ the altitudes responsible for the position of the singularities (note that the left endpoint of X starts at a singularity, and the vertical direction does not have any saddle connections, thus it gets mapped to X without hitting another singularity). We zip neighboring rectangles R_j and R_{j+1} from the bottom upto the height a_j and then split them. To complete the zipping, neighboring rectangles $R_{\pi^{-1}(j)}$ and $R_{\pi^{-1}(j+1)}$ are zipped from the top down to the height $h_{\pi^{-1}(j)} - a_{\pi^{-1}(j)}$.

The parameters h_j and a_j are not independent, they satisfy certain linear equations and linear inequalities (the picture zips up completely, and the zippers do not exceed the top of the rectangles). Continuously varying the parameters λ, h, a while respecting the linear relations yields coordinates in the stratum. Veech showed that given an interval exchange transformation without saddle connections, one can always construct a translation surface M and a horizontal segment $X \subset M$ such that that the first return map of the vertical flow to X gives the initial interval exchange transformation. Moreover, there is a n -dimensional family of such flat surfaces, suspensions over the interval exchange transformation. The continuous variation is a completion of the procedure, it includes directions with saddle connections, we will refer to the completion as the *boundary* of the space of zippered rectangles.

The choice of X completely determines the decomposition, thus one would like a canonical way to choose X . There is an almost canonical way to choose X , namely place the left endpoint of X at one on the singularities and choose the length of X in such a way that X is the shortest interval satisfying Convention 2 and the length of X is at least 1. We will not need to use it, but we remark that there is a constructive way to find X , start with a sufficiently long interval X have its left end point at a singular point and satisfying Convention 2. Apply the “slow” Rauzy induction as long as the resulting interval has length at least one. For almost all flat surface this procedure will stop after a finite number of steps, yielding the desired interval X .

Since M has a finite number of singularities, and each singularity has a finite number of horizontal prongs, this yields $\sum_{i=1}^n (d_i + 1) = 2g - 2 + n$ choices for X . Thus the space of zippered rectangles can be essentially viewed as a ramified covering of the corresponding connected component $\mathcal{H}^{comp}(d_1, \dots, d_n)$ of the stratum. Supposing that the area of the surfaces is one, defined by the condition $\lambda \cdot h = 1$ (a final linear equation), yields the space of zippered rectangles of area one covering the space $\mathcal{H}^{comp}(d_1, \dots, d_n)$ of translation surfaces of area one. Slightly abusing notation, we will refer to the space of zippered rectangles as $\mathcal{H}(d_1, \dots, d_n)$. Summarizing, we have shown that the space $\mathcal{H}(d_1, \dots, d_n)$ is the product of a discrete countable parameter, the permutation (Rauzy class) and the closure of an open subset of \mathbb{R}^{3N} (given by the linear equations and linear inequalities), yielding

Lemma 3. *Each stratum $\mathcal{H}(d_1, \dots, d_n)$ is a separable, metric Baire space.*

Note that if we have a translation surface M and a direction θ which decomposes M into a single cylinder, then this direction on this surface is in the boundary of the space of zippered rectangles since the direction of the flow contains saddle connections. However, for *nearby* directions θ' , the pair (M, θ') will appear in the interior of the space of zippered rectangles (there are only countably many directions which admit saddle connections in any translation surface). See Figure 5, left.

Lemma 4. *Within each stratum $\mathcal{H}(d_1, \dots, d_n)$, there is a countable, dense set of (M_m, α_m) , where M_m is compact, square-tiled, and has a single-cylinder direction.*

Proof. Lemma 18 and Remark 7 of [10] show that the set of compact surfaces with single-cylinder directions is dense in each stratum, where denseness is not in the set of zippered rectangles but in the sense of closeness of Abelian differentials. The constructed surfaces have the vertical direction as single-cylinder (which is forbidden in the space of zippered rectangles); we denote this direction as $\hat{\theta}$.

By making arbitrarily small changes to the vectors of the vertical saddle connections, we ensure that the lengths of all vertical saddle connections are rational multiples of the width of the cylinder without changing the fact that the vertical flow is a single-cylinder direction. By rescaling the width by an arbitrarily small amount, we may ensure all such lengths are in fact rational. We then need only translate all the charts of M by an arbitrarily small amount to ensure that all singularities project to rational coordinates; the surface is now square-tiled with a vertical single-cylinder direction.

Periodic directions correspond to rational directions (now that the singularities all project to rational coordinates), so the flow in any irrational direction will be completely aperiodic. By choosing a direction arbitrarily close to vertical, but still irrational, and labeling this direction as α_m , we obtain our countable, dense set. See Figure 4 to see the effect of this minor shear in direction.

In the language of zippered rectangles, we may make arbitrarily small changes to the lengths of the intervals of the base so that the interval exchange is purely periodic, and without loss of generality we may assume that all lengths are rational. By also forcing the heights of the rectangles to be rational, the surface is square-tiled. While this change forbids the surface from the space of zippered rectangles in the vertical direction, which now has saddle connections, an arbitrarily small change to the direction of the flow will remove all saddle connections, so *some* representation of this surface appears in the space of zippered rectangles. \square

We will have no further explicit use for the directions α_m ; their inclusion is a formality to ensure that our surfaces are proper zippered rectangles. We will generally be more concerned with the single-cylinder direction(s) on M_m . In this case we will represent single-cylinder directions as ‘vertical’ in figures for convenience. As such, we will refer to the zippered rectangle (M_m, α_m) simply as M_m . Let $\{M_m\}$ be the countable, dense set of surfaces guaranteed by Lemma 4.

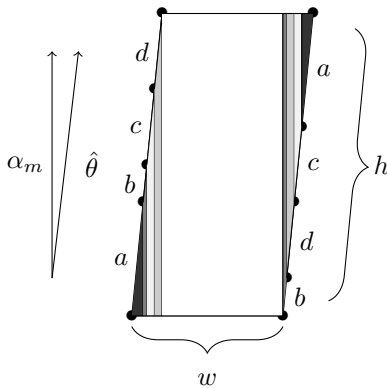


FIGURE 4. The zippered rectangle representation where the vertical direction α_m has no saddle connections and is close to the original single-cylinder direction $\hat{\theta}$. Parallel saddle connections of equal lengths are identified by the letters a through d , and the top and bottom are identified via an interval exchange.

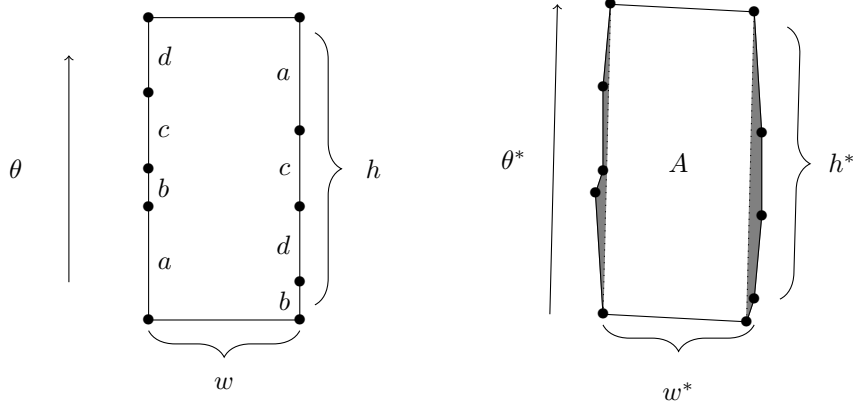


FIGURE 5. Left: a single-cylinder direction θ (not necessarily close to α_m) on the surface M_m with width w and height h . Parallel saddle connections of equal lengths are identified by the letters a through d , and the top and bottom are identified. Right: the vectors corresponding to the saddle connections have been allowed to change slightly; such adjustments give a local metric within each stratum. The flow in direction θ^* (which is very close to θ) will be periodic with nearly the same period h^* on a set A of large measure (excluding the shaded region).

Comparing the surfaces in Figure 5, the new surface is nearby within the same stratum. The flow in direction θ was periodic for the original surface, and a slight adjustment to the direction of the flow in the new surface will yield a cylinder which is of large measure, periodic in a direction very close to θ , with nearly the same period. These claims (direct from Figure 5) are summarized for convenience in the following lemma.

Lemma 5. *Let M_{m_0} be some particular surface taken from the countable, dense subset given by Lemma 4, with a single-cylinder direction given by θ . Then for any $\epsilon > 0$, there is some $\delta > 0$ (depending not just on ϵ but also on the choice of θ) such that for any $(M', \alpha') \in B_\delta(M_{m_0}, \alpha_{m_0})$, we have some direction θ^* on M' for which there is a set $A \subset M'$ with the following properties:*

- *A is a periodic cylinder in direction θ^* , of period h^* and width (measured perpendicular to θ^*) w^* ,*
- *$\mu(A) > (1 - \epsilon)\mu(M')$,*
- *$|h - h^*| < \epsilon \min\{h, h^*\}$ and $|w - w^*| < \epsilon \min\{w, w^*\}$,*
- *$\max\{h, h^*\}|\theta - \theta^*| < \epsilon \min\{w, w^*\}$.*

That is, the surface $(M', \alpha') \in \mathcal{H}(d_1, \dots, d_n)$ has a direction θ' which is very nearly a single-cylinder direction: the representation of the flow in direction θ' is closely approximated by the single-cylinder direction θ on the surface M_{m_0} .

We now are ready to state our main result:

Theorem 1 (Residual Generic Ergodicity). *For each $\mathcal{H}(d_1, \dots, d_n)$, \mathcal{E} is residual within $\mathcal{H}(d_1, \dots, d_n)$: there is a dense G_δ set of surfaces for which the skew product formed with almost-every collection of cuts is ergodic in almost-every direction.*

3. PROOF OF RESIDUAL GENERIC ERGODICITY

Lemma 6. *Let $\{\mathcal{P}, \mu\}$ be a σ -finite measure space, and let Y be a separable metric Baire space such that $\{y_i\}$ ($i = 1, 2, \dots$) is dense in Y . Assume that there is a measurable function*

$$\epsilon(i, k, \psi) : \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathcal{P} \longrightarrow \mathbb{R}^+.$$

Define the sets

$$A_\psi = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{\infty} B_{\epsilon(i, k, \psi)}(y_i),$$

where $B_\epsilon(y) \subset Y$ is the open ball of radius ϵ centered at y . Then the set

$$A = \{y \in Y : \mu(\{\psi \in \mathcal{P} : y \notin A_\psi\}) = 0\}$$

is residual.

Proof. Note that the A_ψ are dense G_δ sets. Without loss of generality, we may assume that $\mu(\mathcal{P}) = 1$; if the result holds in this circumstance, then the assumption that $\{\mathcal{P}, \mu\}$ is σ -finite and the fact that a countable intersection of residual sets in a Baire space is residual completes the proof. It then suffices to prove that for all $\delta > 0$, the set

$$A(\delta) = \{y \in Y : \mu(\{\psi \in \mathcal{P} : y \in A_\psi\}) \geq 1 - \delta\}$$

is residual; if this is shown, then we may intersect $A(1/n)$ for $n = 1, 2, \dots$

So let $\delta > 0$. For each i and k , define $\epsilon^*(i, k) > 0$ so that

$$\mu(\{\psi \in \mathcal{P} : \epsilon(i, k, \psi) < \epsilon^*(i, k)\}) \leq \delta \cdot 2^{-k}.$$

Then the set

$$A^* = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{\infty} B_{\epsilon^*(i, k)}(y_i)$$

is a dense G_δ set. Let $y \in A^*$. Then for each k , there is some i so that $y \in B_{\epsilon^*(i, k)}(y_i)$, which implies that

$$\mu(\{\psi \in \mathcal{P} : y \notin B_{\epsilon(i, k, \psi)}(y_i)\}) \leq \delta \cdot 2^{-k}.$$

By intersecting the above over $k = 1, 2, \dots$, we have $\mu(A^*) \geq 1 - \delta$, or $A^* \subseteq A(\delta)$. \square

That is, if for almost every parameter ψ the set A_ψ is residual, then there is a residual set of y which belong to almost-every A_ψ . It is quite important that the set $\{y_i\}$ (the centers of the balls) be invariant with respect to different $\psi \in \mathcal{P}$; otherwise, the result may not hold. For example, one may readily construct second-countable Y and $\{\mathcal{P}, \mu\}$ a probability space so that for almost every $\psi \in \mathcal{P}$ there is some associated residual set $A_\psi \subset Y$, but for *every* $y \in Y$, $\mu(\{\psi \in \mathcal{P} : y \in A_\psi\}) = 0$.

Our σ -finite parameter space in applying Lemma 6 will be

$$\mathcal{P} = \mathbb{S}^1 \times \Gamma,$$

where Γ was given in (1). A point in this space corresponds to a choice of direction θ for geodesic flow $\tilde{\varphi}_t$, combined with a collection of cuts $\gamma \in \Gamma$. Our Baire space Y will be a stratum of translation surfaces (Lemma 3).

Let the stratum $\mathcal{H}(d_1, \dots, d_n)$ be fixed, and denote by $\{M_m\}$ the countable, dense collection of square-tiled translation surfaces which admit some single-cylinder direction from Lemma 4.

Lemma 7. *For almost-every direction θ , for each M_m there is a sequence $\{\theta_n\}$ of single-cylinder directions of period h_n and width w_n on M_m such that*

$$(4) \quad \lim_{n \rightarrow \infty} \frac{h_n}{w_n} \|\theta_n - \theta\| = 0.$$

Proof. As there are only countably many M_m , it suffices to prove the claim for a single M_m , which follows from [13, Lemma 2.6]. \square

We recall [13, Def. 3.1] and strengthen:

Definition 8. Let M be a fixed compact translation surface which has a dense set of single-cylinder directions, let θ be some direction, and let D be a finite subset of M . We will say that the set D is *uniformly self-avoiding* in direction θ if there is a sequence of single-cylinder directions $\theta_n \rightarrow \theta$ such that for each $d \in D$ there is a subsequence θ_{n_j} , along which we may place d at the center of an invariant subset of the single cylinder, the complement of which tends towards zero measure, which contains no singularities of M and no other points of D . See Figure 6. The two sides of this set, separated by the orbit of d , are called the *left* and *right* halves of the *orbit neighborhoods* of d .

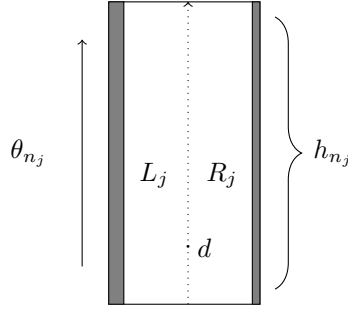


FIGURE 6. By assumption the only element of the set D not within the gray region is the point d , which may then be placed in the center of a sub-cylinder of large measure which avoids all singularities and all other elements of D . As $j \rightarrow \infty$ the grayed-out region tends to zero measure, and thus d tends towards the center of the cylinder. The remaining sub-cylinders are called orbit neighborhoods of d , with left and right halves L_j and R_j .

Lemma 9. *For any $N \in \mathbb{N}$, almost every choice of N points in the plane lifts to every M_m as a uniformly self-avoiding set. Furthermore, if M_{m_0} is fixed, and θ is an irrational direction which admits a sequence $\{\theta_n\}$ of single-cylinder directions (with periods h_n) which satisfies Equation 4, then for each d in this self-avoiding set, if we pass to the subsequence n_j from Definition 8 and denote the left and right halves of a sequence of orbit neighborhoods corresponding to d by L_j and R_j , then we have subsets $L_j^* \subset L_j$, $R_j^* \subset R_j$ such that as $j \rightarrow \infty$,*

- $\mu(L_j^*) \rightarrow (1/2)\mu(M_{m_0})$,
- $\sup_{x \in L_j^*} \|\varphi_{h_{n_j}}^\theta(x) - x\| \rightarrow 0$,
- for any fixed $t_0 \in \mathbb{R}^+$, $\mu(\varphi_{t_0}^\theta L_j^* \triangle L_j^*) \rightarrow 0$,

and similarly for R_j^* .

Proof. Note that the flows are in direction θ , not θ_{n_j} . For the first claim, as there are countably many M_m , it suffices to prove that almost every choice of N points lifts to any particular M_{m_0} as a uniformly self-avoiding set. We rescale a fixed M_{m_0} to be a k -fold cover of the unit square and denote the affine image of θ_n by p_n/q_n (periodic directions on covers of the unit square must be rational). Single cylinders therefore have width $1/q_n$ and height kq_n for some fixed $k \in \mathbb{Z}^+$. The first claim now follows from [11, Thm 4.1]: for *any* sequence $q_n \rightarrow \infty$, for almost every x we have the sequence $kq_n x$ uniformly distributed modulo one. So if there are N points d_1, d_2, \dots, d_N , let $\pi_x(d_i)$ be the x -coordinate of the projection to the plane. So for each $i' \in \{1, 2, \dots, n\}$ we can create a subsequence p_{n_j}/q_{n_j} such that modulo 1 $q_{n_j}\pi_x(d_{i'})$ tends towards $1/2$ as $j \rightarrow \infty$ (and therefore $d_{i'}$ is arbitrarily close to the center of the cylinder) while modulo 1 every other $i \neq i'$, $q_{n_j}\pi_x(d_i)$ tends to either zero or one (and therefore all other d_i are arbitrarily close to the boundary of the cylinder).

Now let d be an element of a uniformly self-avoiding set. We form the sets L_j^* and R_j^* by removing a small neighborhood of the orbit of d (in the direction θ). See Figure 7. Using Equation 4 and Definition 8, we see that L_j^* and R_j^* will tend to half of the total measure. That the sets are nearly invariant under the flow in direction θ (pointwise through time h_{n_j} and as a set through any fixed time t_0) follows from Equation 4 and the fact that h_{n_j} is diverging, respectively: the flow for any fixed t_0 will produce a negligible change in either L_j^* or R_j^* . \square

Compare to [13, Cor. 3.4]. In that result only a specific square-tiled surface was considered, so any positive measure for the sets L, R sufficed. We have strengthened that approach to provide *the same positive limiting measure* for the sets L_j^* and R_j^* for every M_m .

Corollary 10. *For almost every collection of cuts $\gamma \in \Gamma$, the lift of γ to each M_m is unique, and for almost every direction θ the starting points x_i^\pm of the segments of the lift of γ are a uniformly self-avoiding set in the direction θ on every M_m .*

Proof. Uniqueness of the lift of γ to each M_m merely requires that the geodesic from p' , the projection of the marked point $p \in M_{m_0}$, to each (x_i^\pm, y_i^\pm) , as well as the line segments originating from these points, not encounter a singularity when lifted to M . As the set of singularities is of zero measure (it is finite), generic cuts lift uniquely to *each* M_m .

It follows from [12, 14] (see also [13, Lemma 2.6]) that for any fixed M_{m_0} , almost every θ admits a sequence $\{\theta_n\}$ as in Lemma 9, so for each M_m , for almost every θ we have that almost every collection of cuts lifts to M_m uniquely (as outlined above) and with endpoints which are a uniformly self-avoiding set in direction θ . Fubini's theorem allows us to rephrase: almost every collection of cuts lifts uniquely as a self-avoiding set to any fixed M_{m_0} in almost-every direction. Since the collection $\{M_m\}$ is countable, it follows that almost every collection of cuts lifts to a uniformly self-avoiding set on every M_m in almost every direction. \square

Definition 11. We call the pair (γ, θ) *generic* if it satisfies the conclusion of Corollary 10.

To summarize our progress so far: we have a countable, dense collection of surfaces M_m in each stratum. For almost-every direction θ , each one of these

surfaces has a sequence of single-cylinder directions converging in a very strong manner to θ . For almost-every collection of cuts, for each starting point of each cut, on each M_m we may pass to a subsequence of these directions which isolates that starting point from the orbits of every other starting point in a uniform manner. We will now show how this isolation of starting points of the cuts leads to ergodic skew products.

An *essential value* of the skew product $\{\tilde{M}, \tilde{\mu}, \tilde{\varphi}_t\}$ is a $g \in G$ such that for any set $A \subset M$ of positive measure and $\epsilon > 0$ there is a $t > 0$ such that (recall (3))

$$\mu(A \cap \varphi_t(A) \cap S_t^{-1}(B_\epsilon(g))) > 0.$$

The set of essential values forms a closed subgroup of G , and the skew product is ergodic if and only if the flow $\{M, \mu, \varphi_t\}$ is ergodic and the set of essential values is all of G (see e.g. [1]). Clearly, for $\{\tilde{M}, \tilde{\mu}, \tilde{\varphi}_t\}$ to be ergodic it is necessary that the values f takes on the cuts to generate a dense subgroup of G . Our technique for finding essential values is the following:

Lemma 12. *Suppose that M is a translation surface with geodesic flow $\{M, \mu, \varphi_t\}$ in some direction θ , with a collection of cuts γ and corresponding skew product $\{\tilde{M}, \tilde{\mu}, \tilde{\varphi}_t\}$. For $j = 1, 2, \dots$, assume we have times $t_j \rightarrow \infty$ and sets L_j^* such that*

- L_j^* contains no singularities of M and no endpoints of any cut,
- $\lim \mu(L_j^*) = (1/2)\mu(M)$,
- $\sup_{x \in L_j^*} \|\varphi_{t_n}^\theta(x) - x\| \rightarrow 0$,
- for any fixed $t_0 \in \mathbb{R}^+$, $\mu(L_j^* \triangle \varphi_{t_0}^\theta L_j^*) \rightarrow 0$, and
- for all $x \in L_j^*$, $S_{t_j}(x) = f(\gamma_i)$.

Then the skewing values $\pm g_i = f(\gamma_i^\pm)$ taken on the cuts γ_i^\pm are essential values of $\{\tilde{M}, \tilde{\mu}, \tilde{\varphi}_t\}$.

Proof. Note that we have required the ergodic sums $S_{t_j}(x)$ to be constant on each L_j^* and R_j^* , which each form a sequence of *quasi-rigidity sets* [13, Def. 2.1], and we directly have each of $\pm g_i$ as essential values by applying [13, Lemma 2.2]. \square

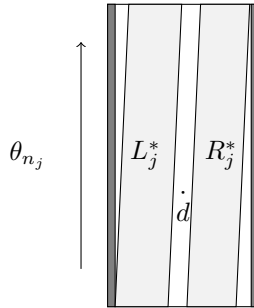


FIGURE 7. The flow in direction θ is slightly skewed compared to direction θ_{n_j} , but sufficiently close to allow for the construction of the sets L_j^* and R_j^* , each of measure very nearly one half the total measure.

Compare to Lemma 9 and Figure 7: the last condition in Lemma 12 is the last remaining condition to verify in order to begin producing essential values for our skew products.

Lemma 13. *Fix generic $(\gamma, \theta) \in \Gamma \times \mathbb{S}^1$, and let $f : \gamma \rightarrow G$, where G is a locally compact Abelian group. Then for each M_{m_0} we have that for each γ_i^\pm with value $f(\gamma_i^\pm)$ we may pass to a subsequence $\{\theta_{n_j}\}$ of single cylinder directions so that we may find a sequence of sets L_j^* to satisfy Lemma 12.*

Proof. The proof is a slightly more refined application of the same use of [11, Thm. 4.1] that we used in Lemma 9. Pick a generic (γ, θ) , and let M_{m_0} be fixed. From Lemma 7 we obtain a sequence θ_n of single-cylinder directions on M_{m_0} converging to θ . For each pair of cuts γ_i^\pm , in direction ζ_i and with length ℓ_i , project the length of γ_i in the direction θ and denote this projected length $\pi(\ell_i)$. Let w_n be the width of the single-cylinder on M_{m_0} in direction θ_n .

Pick a particular pair of cuts $\gamma_{i'}^\pm$. From Lemma 9, let us assume without loss of generality that our sequence θ_n already isolates the starting point $\mathbf{x}_{i'}^+$ of $\gamma_{i'}^+$ from the starting point of every other cut (including $\mathbf{x}_{i'}^-$). For each $i \neq i'$ (and $\mathbf{x}_{i'}^-$), we pass to successive subsequences to guarantee that the *endpoints* of the cuts γ_i^\pm are also forced to be near the boundary of the cylinders. More precisely we require that $(\pi(\ell_i)/w_{n_j}) \bmod 1$ converges to zero/one as $j \rightarrow \infty$, i.e. the cuts γ_i^\pm cross the cylinders very nearly an exact integer number of times. As the projected length $\pi(\ell_i)$ is equal for both γ_i^+ and γ_i^- this subsequence automatically isolates the endpoint of γ_i^- near the boundary as well. Since there are only finitely many $i \neq i'$, we may presume, then, that *every* cut other than $\gamma_{i'}$ has this property: both endpoints of each γ_i^\pm are trapped near the boundaries of the cylinders.

So now consider the chosen pair of cuts $\gamma_{i'}^\pm$, with starting points $\mathbf{x}_{i'}^\pm$. Since we began with a subsequence of directions according to Definition 8, we have that $\mathbf{x}_{i'}^+$ is placed near the centers of the cylinders, and $\mathbf{x}_{i'}^-$ is trapped near the boundary along with all the other starting points. Since the length $\ell_{i'}$ of this particular pair of cuts was chosen generically, by passing to a subsequence once more we may force the *endpoint* $\varphi_{\ell_{i'}}^{\zeta_{i'}}(\mathbf{x}_{i'}^+)$ to be trapped near the *boundary* of the cylinder. Equivalently, we may generically force $(\pi(\ell_{i'})/w_{n_j}) \bmod 1$ to converge to $1/2$: the length of this cut traverses the cylinders very nearly an integer number of times, *plus one half*.

So for $i \neq i'$, the cuts begin very near the boundary of the single cylinders and are of length (projected to direction θ , which is very near to the directions θ_{n_j}) very nearly an integer times the width of the cylinder. Away from the boundary of the cylinder, then, *the ergodic sums of any point flowing the height of the cylinder see no contribution from the cut $\gamma_{i'}^\pm$!* As the pair of cuts $\gamma_{i'}^\pm$ are of equal length, they must traverse the cylinder an equal number of times. Since we have $f(\gamma_{i'}^-) = -f(\gamma_{i'}^+)$, the net effect is that these cuts do not contribute to the ergodic sums for most points in this direction through the height of the cylinder.

For the remaining cut $\gamma_{i'}^\pm$, the endpoint of $\gamma_{i'}^+$ is very near the boundary of the cylinder, while the endpoint of $\gamma_{i'}^-$ is very near the *middle* of the cylinder (a reversal of the placement of the starting points $\mathbf{x}_{i'}^\pm$, as the projected length $\pi(\ell_{i'})$ is very nearly an integer-plus-one-half times the width of the cylinder). So we find that the ergodic sums neatly split the cylinder; away from the boundary and away from the middle, the only two values seen are exactly $g_{i'} = f(\gamma_{i'})$ on the right half of

the cylinder and $-g_{i'} = -f(\gamma_{i'})$ on the left. See Figure 8; the sets L_j^* and R_j^* are obtained from this representation in the same way as in Figure 7. \square

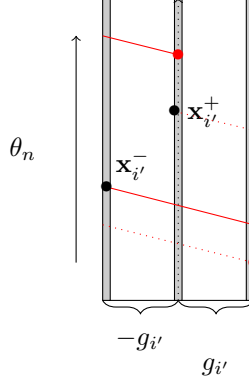


FIGURE 8. For generic cuts and generic directions, on each particular M_{m_0} we may pass to a subsequence of single-cylinder directions on which the cuts traverse the single cylinder in a regular manner. All cuts other than a chosen $\gamma_{i'}^\pm$ have both endpoints restricted near the boundary and therefore contribute nothing to the ergodic sums in the cylinder, while the chosen cut $\gamma_{i'}^\pm$ splits the cylinder into two halves with very precise ergodic sums.

We finally turn to the question of the appropriate ϵ to use as the radius of the balls in constructing our G_δ sets to complete the proof of Theorem 1. Let $\psi = (\theta, \gamma) \in \mathcal{P}$ be generic in the sense that all generic results proved up to this point hold simultaneously (Lemma 7, Lemma 9, Corollary 10, Lemma 12, and Lemma 13).

Let k be a fixed positive integer, and fix some M_{m_0} . For each d , an endpoint of a cut, we have a sequence of single-cylinder directions θ_n according to Lemma 9. We may therefore find *some* single-cylinder direction θ_{n_j} on M_{m_0} which produces two orbit-neighborhoods L_j^* and R_j^* according to Figure 7 such that each has measure at least $(k-1)/2k$ and each is of height at least k . Fix this choice of single-cylinder direction θ_{n_j} (which depends not only on m_0 and θ , but also k).

We let $\epsilon(m_0, k, d) > 0$ be small enough (recall Lemma 5, making the size of the balls in the stratum smaller if necessary) that any surface within $\epsilon(m_0, k)$ of each (M_{m_0}, α_{m_0}) satisfies:

- (1) the lift of γ to all nearby M in this neighborhood will be unique,
- (2) excluding a set of measure at most $\mu(M)/k$, there is a periodic cylinder in direction θ' of period h' such that $\max(h, h') \cdot |\theta_{n_j} - \theta'| < w/(2k)$ for all points in this cylinder, where w is the width of the cylinder. See Figure 5, right.
- (3) for those $x \in M$ of within this large set and also not within distance $w/(2k)$ of any lift of a cut, we have

$$S_{h'}(x, M, \theta) = S_{h'}(x, M, \theta') = S_h(\hat{x}, M_{m_0}, \theta_{n_j}) = S_h(\hat{x}, M_{m_0}, \theta),$$

where \hat{x} denotes the lift to M_{m_0} of the projection of $x \in M$ to the plane, and θ_{n_j} is a single-cylinder direction on M_{m_0} very near θ . Informally, point-by-point these conditions amount to:

- (1) varying the edges of the polygon will not cause any cut to cross a singularity,
- (2) the flows in the two directions θ and θ' through times h or h' are more-or-less indistinguishable for most points, and also virtually indistinguishable from the single-cylinder direction θ_{n_j} , and
- (3) for most of the surface M , the ergodic sum through time h' in direction θ is identical to the ergodic sum through time h in direction θ of the corresponding point on M_{m_0} .

The equality in the last item is accomplished by starting with M_{m_0} in direction θ , approximating by a single-cylinder direction θ_{n_j} , choose a surface in the ϵ -ball around M_{m_0} by making small adjustments to the vectors which define the saddle connection in direction θ_{n_j} to form the surface M , finding a cylinder of large measure on the resulting surface in direction θ' , and then returning the flow to direction θ . At each stage, ergodic sums are maintained pointwise except on sets of arbitrarily small measure, according to the diameter of the ball used.

Having chosen the cuts γ and the direction of the flow θ , we therefore have a small ball around M_{m_0} for which any surface in that ball is very closely approximated by the single-cylinder direction θ_{n_j} on M_{m_0} , at least through time $h_{n_j} \geq k$. As k increases, the strength of this approximation necessarily improves in two ways: increasing height of the cylinders forces $\theta_{n_j} \rightarrow \theta$, and the measure of the orbit neighborhoods which directly transfer to any M in the ball is tending towards $1/2$. So with m_0 fixed, for each k and each endpoint of each cut d , we obtain an allowable radius $\epsilon(m_0, k, d)$. Define $\epsilon(m_0, k, \psi)$ to be the minimum of these radii across the finite number of endpoints of the cuts (recall the original dependence on ψ , the collection of cuts and the direction of geodesic flow).

Let

$$A_\psi = (E(\theta)) \cap \left(\bigcap_{k \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} B_{\epsilon(i, k, \psi)}(M_m, \alpha_m) \right),$$

where $E(\theta)$ is the set of M for which the flow in direction θ is ergodic, and θ was the direction of geodesic flow specified in ψ .

Lemma 14. *Let $M \in A_\psi$, and let $\tilde{M} = M \times G$ with geodesic flow in direction θ defined as the skew product over M given by (2). Then the geodesic flow on \tilde{M} in direction θ is ergodic.*

Proof. The set $E(\theta)$ is residual in each stratum for any θ [9, Corollary 2], so A_ψ is residual as well. For $M \in A_\psi$, then, the flow in direction θ is ergodic. It remains to show that each value $g \in G$ which is the value assigned to some cut is an essential value of the flow $\{\tilde{M}, \tilde{\mu}, \tilde{\varphi}_t\}$; since these values generate a dense subgroup of G , this finding will complete the proof.

Pick an arbitrary $g \in \{g_1, \dots, g_N\}$ (the values taken on the cuts), and let \mathbf{x}_i^+ be the starting point of a cut which takes value g . Create a sequence of approximations as in Figure 9. By construction, as $M \in A_\psi$, we have a sequence of single-cylinder surfaces M_m with corresponding representations as in Figure 9, from which we may pull back to M a set of measure arbitrarily close to $1/2$ on which we have ergodic sums constantly equal to g ; via Lemma 13 the ergodic sums on the corresponding

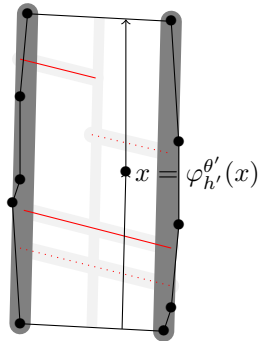


FIGURE 9. If M is sufficiently close to M_{m_0} , then ‘most’ of M looks like a single cylinder (all but the dark-gray region, as in Figure 5 (right) and Figure 6), and the cuts are in almost the same place, with the same number of crossings, so that for ‘most’ of this periodic cylinder (now also excluding the light-gray regions) the ergodic sums are identical to corresponding points on M_{m_0} (as in Figure 8).

sets in M_m are constant, and our balls were picked small enough that most of these sets transfer to M without affecting the ergodic sums. The value g , then, is an essential value of the flow in direction θ on M (Lemma 12). Since the choice of g was arbitrary and we assumed that $\{g_1, \dots, g_N\}$ generate a dense subgroup of G , it follows that the flow on \tilde{M} in direction θ is ergodic. \square

So for any generic choice of direction θ and cuts γ , we form a residual set of surfaces M for which the skew product transformation on \tilde{M} is ergodic in direction θ . By applying Lemma 6, this completes the proof of Theorem 1.

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SUNY COLLEGE AT OLD WESTBURY, MATHEMATICS/CIS DEPARTMENT, P.O. BOX 210, OLD WESTBURY, NY 11568, USA

E-mail address: `ralstond@oldwestbury.edu`

AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M, UMR 7373, 13453 MARSEILLE, FRANCE

Address: I2M, Luminy, Case 907, F-13288 Marseille CEDEX 9, France

E-mail address: `serge.troubetzkoy@univ-amu.fr`